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GENERALIZATION OF BÉZOUT MODULES

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Abstract

Let R be a commutative ring with identity. In this paper, we generalize the concept of Bézout modules to P-Bézout modules. A module M over R is said P-Bézout if every finitely generated prime submodules N of M is cyclic. We give an example of P-Bézout module which is not a Bézout module.

1. Introduction

The ring which considered in this paper is commutative with identity. A module M over R is said *Bézout* if every finitely generated submodules N of M is cyclic [1].

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From [2], we have the concept of *P*-Bézout ring. Based on that concept, we generalize the concept of Bézout module to *P*-Bézout module as follows.

Definition 1.1. A module M over R is said *P-Bézout* if every finitely generated prime submodules N of M is cyclic.

Of course, every Bézout module is a *P*-Bézout module.

2. Result

In this section, it will be discussed examples of *P*-Bézout module. Let us consider the following example.

Example 2.1. Consider rational number $\mathbb{Q}$ as a $\mathbb{Z}$-module. Gaur et al. (see in [3]) proved that the prime submodules of $\mathbb{Q}$ over $\mathbb{Z}$ is only 0 and 0 is finitely generated. We will give the detail proof and, by definition, prove that its module is a *P*-Bézout module.

Let N be a prime submodule of $\mathbb{Q}_{\mathbb{Z}}$. Set $J = [N : \mathbb{Q}]$ as an ideal of $\mathbb{Z}$, so that $J\mathbb{Q} = [N : \mathbb{Q}]\mathbb{Q} \subseteq N$. Now let A and B are ideals of $\mathbb{Z}$ such that $AB \subseteq J$. It yields $A(B\mathbb{Q}) = AB\mathbb{Q} \subseteq J\mathbb{Q} \subseteq N$. Since N is a prime submodule, $A \subseteq J$ or $B\mathbb{Q} \subseteq N$. By the same argument, it yields $B \subseteq J$. Therefore, J is a prime ideal of $\mathbb{Z}$.

Let $x \in J$, so we have $x\mathbb{Q} \subseteq N$. Since $N \neq \mathbb{Q}$, it follows $x\mathbb{Q} \subset N$. Since $x\mathbb{Q} \subset N$ and $\mathbb{Q}$ is a field, $x = 0$ and it follows that $J = 0$.

Suppose $N \neq 0$. Since N is prime submodule, there are nonzeros $a, b \in \mathbb{Z} \subseteq \mathbb{Q}$ such that $ab^{-1} \in N$. But $a \notin J$, then $b^{-1} \in N$ and it follows that $1 \in N$. So we have $\mathbb{Z} \subseteq N$.

And N is also a pure submodule of $\mathbb{Q}$. So, there are nonzeros $\alpha, \beta \in \mathbb{Z}$ such that $\alpha\beta^{-1} \in \mathbb{Q}$ but $\alpha\beta^{-1} \notin N$. We know that $\beta\alpha\beta^{-1} = \alpha \in \mathbb{Z} \subseteq N$. Since N is prime submodule and $\beta \notin J$, $\alpha\beta^{-1} \in N$. This is a contradiction. So it follows that $N = 0$.

Since prime submodules of $\mathbb{Q}$ over $\mathbb{Z}$ is only 0 and 0 is finitely generated and cyclic, so $\mathbb{Q}$ as a $\mathbb{Z}$-module is a *P*-Bézout module.

Example 2.2. Consider *R* is a field and $M = R^2$. It is clear that *M* is a two dimensional vector space and finitely generated prime submodules are every subspace. It follows that *M* is *P*-Bézout module since as we know that every subspace is a field, hence it is cyclic over *R*. But, *M* is not Bézout module since there exists *M* is finitely generated but it is not cyclic.

Example 2.3. Let see another example of a *P*-Bézout module which is not a Bézout module. Consider $M = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}\right\}$ as an *R*-module. Submodules of *M* are just $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ and *M* itself. As we know that finitely generated prime submodules of *M* just are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$. Since they are cyclic submodules, *M* is a *P*-Bézout module. It is clear that *M* is finitely generated but not cyclic. So *M* is not a Bézout module.

Example 2.4. Consider *R* is a ring and $M = R/I \oplus R/J$ with *I* and *J* is a different maximal ideal of *R*, is a module over *R*. It is easy to show that *M* is *P*-Bézout but not Bézout.

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