Far East Journal of Mathematical Sciences (FJMS)

Published Online: June 2016
http://dx.doi.org/10.17654/MS099111723
Volume 99, Number 11, 2016, Pages 1723-1732 **ISSN: 0972-0871**

A NOTE ON BÉZOUT MODULES

Muhamad Ali Misri[1,2], Hanni Garminia[2], Irawati[2] and Pudji Astuti[2]

[1]Department of Mathematics Education
IAIN Syekh Nurjati Cirebon
Jl. By Pass Perjuangan Kesambi
Cirebon, Indonesia
e-mail: alimisri@syekhnurjati.ac.id
alimisri@gmail.com

[2]Department of Mathematics
Institut Teknologi Bandung
Jl. Ganesa 10 Bandung 40132
Indonesia

Abstract

All rings are commutative with identity and all modules are unital. In this paper, we investigate submodules of Bézout modules. The results obtained are then used to give some properties of faithful Bézout modules. We also give several properties of multiplication Bézout modules.

1. Introduction

Let R be a commutative ring with identity and M be a unital R-module. An R-module M is called *Bézout* if every finitely generated submodule N of M is cyclic, see [1]. Definition of Bézout ring is obtained by viewing R as an

Received: January 25, 2016; Revised: March 14, 2016; Accepted: April 12, 2016
2010 Mathematics Subject Classification: 16D80, 13C05, 13C13, 13C99.
Keywords and phrases: Bézout modules, Bézout rings, multiplication modules.
Communicated by K. K. Azad

R-module, that is, a ring which is every finitely generated ideal I of R is principal. The properties of Bézout modules can be obtained by relating these modules to other modules. Hence the study of relationship between Bézout modules and other modules is needed. Some properties for Bézout modules can be found in [1-3] and for Bézout rings can be found in [5, 7, 9-12, 15-17, 20].

For any submodule N of M, $[N : M]$ denotes the annihilator of M/N, that is, $[N : M] = \{r \in R \mid rM \subseteq N\}$. A proper submodule N of M is called *prime* if $rm \in N$ for some $r \in R$ and $m \in M$ implies that either $r \in [N : M]$ or $m \in N$, see [14]. Equivalently, $[N : K] = [N : M]$ for every submodule $N \subseteq K \subseteq M$, see [13, p. 765]. Equivalently, $IK \subseteq N$ for some ideal I of R and submodule K of M implies that either $I \subseteq [N : M]$ or $K \subseteq N$, see [8].

An R-module M is called *multiplication* if for each submodule N of M there exists an ideal I of R such that $N = IM$. Note that this is equivalent to $N = [N : M]M$, see [6]. It is clear that every cyclic R-module is a multiplication module, see [6, p. 175].

Let M be a multiplication R-module. It can be proved that a proper submodule N of M is prime if and only if the ideal $[N : M]$ of R is also prime. The detailed proof appears in [13]. It is also shown that if each prime submodule of M is finitely generated, then M is Noetherian. This was stated by Behboodi and Koohy [8, Corollary 3]. The same result could be found in Gaur et al. [14, Theorem 3.2].

An R-module M is called *faithful* if $[0 : M] = 0$. Let N be a submodule of a faithful multiplication R-module M. Smith [21, Prop. 13] has proved that $g(N) \leq g([N : M])g(M)$ and $g([N : M]) \leq g(N)g(M)$. Here $g(M)$ denotes the least positive integer n such that M can be generated by n elements if M is finitely generated and ∞ otherwise. An R-module M is called a *cyclic submodule module* (*CSM*) if every submodule of M is cyclic.

An integral domain D is called a *principal ideal domain* (*PID*) if every ideal I of D is principal. It is showed that a faithful multiplication module M over an integral domain D is a CSM if and only if D is a PID (see [1] for more details).

In this paper, we investigate other properties of a Bézout module. Our interest in this investigation came as a result of our attempt to find a relationship between Bézout modules and other modules.

In fact, Ali mentions that obviously a CSM is a Bézout module. Thus, every faithful multiplication module over a PID is a Bézout module (see [1]). Moreover, he also proved that a faithful multiplication module M over an integral domain R is a Bézout module if and only if R is a Bézout domain (see [1, Proposition 1.2]).

However, generalizing Proposition 1.2 in [1] via extending integral domain to arbitrary ring gives that if M is a faithful Bézout R-module, then R is a Bézout ring (see Theorem 1) and if M is a faithful cyclic module over Bézout ring, then M is Bézout (see Theorem 2). The another result gives that if M is a multiplication Bézout R-module in which every prime submodule is finitely generated, then M is a CSM (see Proposition 2). These give a relationship between multiplication modules, faithful modules and Noetherian modules. Some relevant examples and counterexamples are indicated.

2. Faithful Bézout Modules

In this section, we investigate submodules of Bézout modules. The results obtained are then used to give some properties of faithful Bézout modules, which were stated in Theorem 1.

As for prerequisites, the reader is expected to be familiar with the idealization of a module (see [3-5] and [19]). Using this concept, we will construct examples of Bézout modules. Here are definition of an idealization and required properties. Let M be an R-module. The direct sum

$$R \oplus M = \{(r, m) \mid r \in R, m \in M\}$$

with addition and multiplication operations, respectively, given by $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ is a commutative ring with identity. This ring is called an *idealization of R-module*, denoted by $R_{(+)}M$. It first was introduced by Nagata in [4].

Let I be an ideal of R and N be a submodule of M. It is obvious that not all subsets $I_{(+)}N$ of $R_{(+)}M$ are ideal. For example, consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. The subset $3\mathbb{Z}_{(+)}0$ of the idealization $\mathbb{Z}_{(+)}\mathbb{Q}$ is not an ideal.

Let M be an R-module, I be an ideal of R and N be a submodule of M. In [5], Anderson and Winders stated that if $IM \subseteq N$, then $I_{(+)}N$ is an *ideal* of $R_{(+)}M$. An ideal H of $R_{(+)}M$ is called *homogeneous* if $H = I_{(+)}N$, where I is an ideal of R and N is a submodule of M. In this case, $I_{(+)}N = (R_{(+)}M)(I_{(+)}N) = I_{(+)}(IM + N)$ gives that $IM \subseteq N$. Every ideal of $R_{(+)}M$ of the form $I_{(+)}N$ is homogeneous. Let see the above-mentioned example. It easy to check that all nonzero ideal of $\mathbb{Z}_{(+)}\mathbb{Q}$ have the form $n\mathbb{Z}_{(+)}\mathbb{Q}$ for all $n \in \mathbb{Z}$ and they are homogeneous. It follows that $\mathbb{Z}_{(+)}\mathbb{Q}$, where its ideal satisfying $n\mathbb{Z}_{(+)}\mathbb{Q} = \mathbb{Z}_{(+)}\mathbb{Q}(n, 0)$, is Bézout.

The following proposition, could be found in [3, Theorem 6], gives relationship between ring and module via idealization.

Proposition 1. *Let M be an R-module and* $R_{(+)}M$ *be an idealization of R-module M. If* $R_{(+)}M$ *is a Bézout ring, then R is a Bézout ring and M is a Bézout module. The converse is true if M is finitely generated and* $R_{(+)}M$ *is homogeneous.*

Proposition 1 is useful for constructing examples. Let us consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Since $\mathbb{Z}_{(+)}\mathbb{Q}$ is Bézout, by Proposition 1, both $\mathbb{Z}$ and $\mathbb{Q}$ are Bézout.

After constructing an example of Bézout module, we need to consider the following trivial lemma. This lemma will be used to prove Theorem 1. Let us now start the investigation for giving other properties of Bézout modules.

Lemma 1. *Every submodule of a Bézout R-module is Bézout*.

For example, view the set of rational numbers $\mathbb{Q}$ as a $\mathbb{Z}$-module. Every sub-module of $\mathbb{Q}$ as a $\mathbb{Z}$-module is Bézout.

Let M be an R-module and N be a submodule of M. If N is a Bézout sub-module of M, then M needs not Bézout. Consider F is a field and $M = F^2$. It is clear that M is a two dimensional vector space. It is easy to check that F as a submodule of M is Bézout. But M is not Bézout, see [18] for more details.

Theorem 1. *Let M be a faithful R-module*. *If M is a Bézout module*, *then R is a Bézout ring*.

Proof. Suppose M is a Bézout faithful R-module. Set $\theta_m : R \to M$ for some $m \in M$ given by $r \mapsto rm$ for each $r \in R$. It is easy to check that the map θ_m is an R-homomorphism. Since $[0 : M] = 0,$ we have $Ker(\theta_m) = \{r \in R | \theta_m(r) = 0\} = \{r \in R | rm = 0\} = 0.$ This gives that the map θ_m is a monomorphism and hence $R \cong R/0 \cong \theta_m(R).$

Since $R \cong \theta_m(R)$ and $\theta_m(R) \subseteq M,$ by Lemma 1, R as an R-module is Bézout. This means that R is a Bézout ring, which completes the proof. □

Let us consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. The module $\mathbb{Q}$ is a faithful module. According to the above example, $\mathbb{Z}$-module $\mathbb{Q}$ is a Bézout module. By using Theorem 1, $\mathbb{Z}$ is a Bézout ring. Conversely, view direct sum $\mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module. It is obvious that $\mathbb{Z} \oplus \mathbb{Z}$ is a faithful module and $\mathbb{Z}$ is a Bézout ring. But, $\mathbb{Z} \oplus \mathbb{Z}$ is not a Bézout module. Another example is a two dimensional vector space M over a field F, that is, $M = F^2$. It is known that the faithful module M is not Bézout even though the field F is Bézout.

Let M be an R-module and $0 \neq v \in M$. Element v is called *torsion* if $rv = 0$ for some nonzero $r \in R$. An R-module M is called *torsion free* if have no nonzero torsion element. It is clear that every torsion free module is faithful.

Corollary 1. *Let M be a torsion free R-module. If M is a Bézout module, then R is a Bézout ring.*

Proof. Suppose M is a Bézout torsion free R-module. According to the above remark, indeed M is also a faithful module. It follows that M is faithful Bézout, by Theorem 1, implies R is Bézout. This is the desired conclusion. □

3. Multiplication Bézout Modules

In this section, we give several properties of multiplication Bézout modules. It is known that every cyclic module is multiplication. Therefore, we will first investigate a relationship between cyclic modules and Bézout modules.

Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. It is known that $\mathbb{Q}$ is a Bézout module but not finitely generated over $\mathbb{Z}$. Thus, not all Bézout modules are cyclic.

Lemma 2. (i) *Every finitely generated Bézout module is cyclic.*

(ii) *Every Noetherian Bézout module is a CSM.*

Proof. (i) This statement is trivial. (ii) Suppose M is a Noetherian Bézout module. Let N be a submodule of M. Indeed, the submodule N of M is finitely generated and hence cyclic by the definition of Bézout module. It follows that M is a CSM, which proves the lemma. □

We now apply Lemma 2 to obtain the proposition below.

Proposition 2. *If M is a multiplication Bézout R-module in which every prime submodule is finitely generated, then M is a CSM.*

Proof. Suppose M is a multiplication Bézout R-module in which every prime submodule is finitely generated. By [14, Theorem 3.2], M is a

Noetherian module. By using Lemma 2(ii), it follows that M is a CSM. This is the desired conclusion. □

Proposition 2 tells us about a Bézout module which is cyclic. Note that not all cyclic modules are Bézout. Consider the following example. View $\mathbb{Z}[x]$ as a $\mathbb{Z}[x]$-module. Thus, ideal $\langle 2, x \rangle$ of $\mathbb{Z}[x]$ as a submodule is not cyclic. It follows that $\mathbb{Z}[x]$ is cyclic but not Bézout. Meanwhile, every CSM is a Bézout module.

Theorem 2. *Let R be a Bézout ring and M be a cyclic R-module. Then the module M is Bézout if it is faithful.*

Proof. Suppose R is a Bézout ring and M is a cyclic *faithful* R-module. This is clear that M is finitely generated multiplication *faithful* module.

Now, let N be a finitely generated submodule of M. It gives the ideal $[N : M]$ is also finitely generated by [21, Prop. 13], since M is finitely generated *faithful* multiplication and $N = [N : M]M$. Moreover, since R is Bézout, the ideal $[N : M]$ is principal. The submodule N could be written as $N = RaRm = Ram$ for some $a \in R$ and $m \in M$. Therefore, N is cyclic and it gives M is a Bézout module. □

Consequence of Theorem 1 and Theorem 2, which is stated in Corollary 2, is an extension of Proposition 1.2 in [1].

Corollary 2. *Let M be a faithful cyclic R-module. Then the ring R is Bézout if and only if the module M is Bézout.*

For example, consider $n\mathbb{Z}$ as a $\mathbb{Z}$-module. It yields $n\mathbb{Z}$ is a cyclic faithful module. Since $\mathbb{Z}$ is Bézout, by Corollary 2, $n\mathbb{Z}$ is also Bézout.

Let M_i $(1 \leq i \leq n)$ be a finite collection of cyclic R-modules and $M = M_1 \oplus \cdots \oplus M_n$. El-Bast and Smith [13, Corollary 2.4] have proved that M is a multiplication module if and only if M is cyclic. Theorem 1 and Theorem 2 together with [13, Corollary 2.4] give the following corollary.

Corollary 3. *Let M be a faithful multiplication R-module and $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ with M_i $(1 \leq i \leq n)$ is a finite collection of cyclic R-modules. Then the ring R is Bézout if and only if the module M is Bézout.*

Proof. Suppose R is a Bézout ring, M is a faithful multiplication R-module and $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ with M_i $(1 \leq i \leq n)$ is a finite collection of cyclic R-modules. By [13, Corollary 2.4] and Theorem 2, M is a Bézout module. Conversely, suppose M is a faithful Bézout R-module. Applying Theorem 1 yields R is a Bézout ring. □

Let R be a commutative ring with identity and M be a multiplication R-module with only finitely many maximal submodules. Then M is cyclic [13, Theorem 2.8]. Next, corollary is given by Theorem 2 together with [13, Theorem 2.8] as follows.

Corollary 4. *Let M be a faithful multiplication R-module with only finitely many maximal submodules. Then the ring R is Bézout if and only if the module M is Bézout.*

Proof. Suppose R is a Bézout ring and M is a faithful multiplication R-module with only finitely many maximal submodules. By [13, Theorem 2.8] and Theorem 2, M is a Bézout module. The converse runs as on the proof of Corollary 3. □

Every Artinian multiplication module is cyclic [13, Corollary 2.9]. Here is the last corollary which is given by Theorem 2 together with [13, Corollary 2.9] as follows.

Corollary 5. *Let M be an Artinian faithful multiplication R-module. Then the ring R is Bézout if and only if the module M is Bézout.*

Proof. Suppose R is a Bézout ring and M is an Artinian faithful multiplication R-module. By [13, Corollary 2.9] and Theorem 2, M is a Bézout module. The converse runs as on the proof of Corollary 3. □

Acknowledgements

This research is supported by Hibah Desentralisasi DIKTI. The authors thank the anonymous referees for their valuable suggestions which let to the improvement of the manuscript.

References

[1] Majid M. Ali, Invertibility of multiplication modules, New Zealand J. Math. 35(1) (2006), 17-29.

[2] Majid M. Ali, Idealization and theorems of D. D. Anderson, Comm. Algebra 34 (2006b), 4479-4501.

[3] Majid M. Ali, Multiplication modules and homogeneous idealization, II, Beiträge Algebra Geom. 48(2) (2007b), 321-343.

[4] D. D. Anderson, Some remarks on multiplication ideals, II, Comm. Algebra 28(5) (2000), 2577-2583.

[5] D. D. Anderson and Michael Winders, Idealization of a module, J. Commut. Algebra 1(1) (2009), 3-56.

[6] A. Barnard, Multiplication modules, J. Algebra 71(1) (1981), 174-178.

[7] R. A. Beauregard and R. E. Johnson, Primary factorization in a weak Bezout domain, Proc. Amer. Math. Soc. 25 (1970), 662-665.

[8] M. Behboodi and H. Koohy, On minimal prime submodules, Far East J. Math. Sci. (FJMS) 6(1) (2002), 83-88.

[9] Willy Brandal, Constructing Bézout domains, Rocky Mountain J. Math. 6(3) (1976), 383-399.

[10] Fatima Cheniour, On Bézout rings, Int. J. Algebra 6(32) (2012), 1507-1511.

[11] P. M. Cohn, Bézout rings and their subrings, Proc. Camb. Phil. Soc. 64 (1968), 251-264.

[12] Mohammed S. El-Atrash and Arwa E. Ashour, On primary compactly packed Bézout modules, Islamic University of Gaza, 2002.

[13] Zeinab Abd El-Bast and Patrick F. Smith, Multiplication modules, Comm. Algebra 16(4) (1988), 755-779.

[14] Atul Gaur, Alok Kumar and Anand Parkash, Prime submodule in multiplication modules, Int. J. Algebra 1(8) (2007), 375-380.

[15] Isaac Goldbring and Marc Masdeu, Bézout domains and elliptic curves, Comm. Algebra 36(12) (2008), 4492-4499.

[16] Dino Lorenzini, Elementary divisor domains and Bézout domains, J. Algebra 371 (2012), 609-619. http://www.math.uga.edu/lorenz/Bézout.pdf.

[17] Warren Wm. McGovern, Bézout rings with almost stable range 1, J. Pure Appl. Algebra 212(2) (2008), 340-348.

[18] Muhamad Ali Misri, Hanni Garminia and Irawati, Generalization of Bézout modules, Far East J. Math. Sci. (FJMS) 72(1) (2013), 131-133.

[19] Masayoshi Nagata, Local rings, Interscience Tracts in Pure and Applied Mathematics, John Wiley and Sons, New York, London, 1962.

[20] K. Samei, Flat submodules of free modules over commutative Bézout rings, Bull. Austral. Math. Soc. 71 (2005), 113-119.

[21] P. F. Smith, Some remarks on multiplication modules, Arch. Math. 50 (1988), 223-235.